Junctions of spin-incoherent Luttinger liquids with ferromagnets and superconductors

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We discuss the properties of a strongly interacting spin-charge separated one-dimensional system coupled to ferromagnets and/or superconductors. Our results are valid for arbitrary temperatures with respect to the spin energy, but require that temperature be small compared to the charge energy. We focus mainly on the spin-incoherent regime where temperature is large compared to the spin energy, but small compared to the charge energy. In the case of a ferromagnet we study spin pumping and the renormalized dynamics of a precessing magnetic order parameter. We find that the interaction-dependent temperature dependence of the spin pumping can be qualitatively different in the spin-incoherent regime from the Luttinger liquid regime, allowing an identification of the former. Likewise, the temperature dependence of the renormalized magnetization dynamics can be used to identify spin-incoherent physics. For the case of a spin-incoherent Luttinger liquid coupled to two superconductors, we compute the ac and dc Josephson currents for a wire geometry in the limit of tunnel coupled superconductors. Both the ac and dc responses contain "smoking gun" signatures that can be used to identify spin-incoherent behavior. Experimental requirements for the observation of these effects are laid out.

I. INTRODUCTION

An exotic and distinctive feature of interacting one-dimensional (1-d) systems is the phenomenon of spin-charge separation where the elementary excitations of the fermionic system are decoupled charge and spin bosonic modes that propagate at different velocities.[1–3] At low energies, Luttinger liquid (LL) theory[4] has been quite successful at describing the properties of one-dimensional systems such as quantum wires[5,6] and nanotubes.[7,8] One of the elegant features of the theory is the ease with which one can obtain exact results, thanks to the quadratic form of the Hamiltonian describing the low-energy properties,

$$H_{1\text{-}d} = \sum_i v_i \int \frac{dx}{2\pi} \left[\frac{1}{K_i} [\partial_x \theta_i(x)]^2 + K_i [\partial_x \phi_i(x)]^2 \right], \quad (1)$$

where $i = \rho, \sigma$ stand for the charge and spin sectors and θ_i and ϕ_i are bosonic fields satisfying $[\theta_i(x), \phi_j(x')] = -i\frac{\pi}{2} \delta_{ij} \operatorname{sgn}(x - x')$ representing charge-/spin-density fluctuations and charge/spin current-density fluctuations, respectively, and v_i are the collective mode velocities. The parameters K_i describe the interactions in the 1-d system. For repulsive interactions $K_\rho < 1$, and for SU(2) symmetry in the spin sector $K_\sigma = 1$. Generally, the theory is applied under the assumption that both the spin and charge modes operate at low energy. But at low densities n, where $r_s = (2na_B)^{-1} \gg 1$ (with a_B as the Bohr radius for the material), strong interactions open up a wide window between the characteristic energies of the charge and the spin sectors by suppressing the spin-exchange energy E_σ while enhancing the charge energy E_ρ. When the temperature T is raised into the window of energy, $E_\sigma \ll k_B T \ll E_\rho$ (where k_B is Boltzmann's constant), the spin sector consists of thermally randomized spins that are no longer described by Eq. (1), while the charge sector effectively becomes a spinless LL. A one-dimensional system in this regime is known as a spin-incoherent Luttinger liquid (SILL).[9]

Several properties of the SILL have already been established theoretically,[9] but the experimental effort has been slowed due to the difficulty of reaching the aforementioned window of energy. Nevertheless, there are experimental indications of this regime in high-quality quantum wires grown with the cleaved-edge technique,[10] in recent split-gate devices,[11] and possibly even in low-density carbon nanotubes.[12] Given the recent progress in manufacturing hybrid devices consisting of superconductors (SCs) and ferromagnets (FMs), and one-dimensional systems such as nanotubes, we propose here that the properties of the SILL be experimentally investigated by studying junctions of superconductors and ferromagnets with a SILL. As we show below, these SILL hybrid systems lead to distinctive behaviors (relative to a LL or noninteracting 1-d system) that will allow the SILL to be identified in various types of transport measurements, and in the dynamics of a coupled ferromagnet.

In early work on transport in the spin-incoherent regime, Matveev[13,14] found that a SILL adiabatically connected to noninteracting leads resulted in a universal reduction in the conductance of a single-channel quantum wire from $2e^2/h$ to e^2/h. This effect has been ascribed to the deactivation of the spin channel, whereby the spin excitations are reflected while the charge ones propagate freely. In a recent related work, the authors studied the problem of a SILL adiabatically connected to two superconductors. It was found that the critical Josephson current through such a SC-SILL-SC system suffers the same fate: the critical value of the Josephson current is halved relative to the LL case.[15] These universal results follow from the assumption of adiabaticity of the contacts between the SILL and the leads/SC, which may not be satisfied in some experimental situations.

In this work we study the transport properties of a SILL in the opposite limit where it is contacted via tunnel junctions to a superconductor and/or a ferromagnet. We will discuss several scenarios that may be implemented experimentally with existing technologies. The central aim of this work is to

contribute to our fundamental understanding of the SILL by addressing its properties in hybrid structures, and to add to the arsenal of experiments that can be carried out to probe it and infer its existence.

Our main results are the following. For the case of a FM tunnel coupled to a SILL, we compute the spin current pumped into the SILL due to a time-dependent magnetization in the ferromagnet. The algebraic form of the expression for the pumped spin current is the same as that for noninteracting fermions, or a Luttinger liquid, but the coefficients appearing have a temperature dependence characteristic of the SILL. We argue that the spin transport is diffusive and under this assumption compute the renormalization (due to the pumped spin current and its backflow) of the Gilbert damping α of the magnetization motion in the FM. This also has a characteristic temperature dependence in the spin-incoherent regime. For the case of a SC tunnel coupled to a SILL, we evaluate the ac and dc Josephson currents in a wire that can be either side coupled (bulk coupled) or end coupled to the SC. We find that the dc Josephson current suffers an exponential decay in real space due to the incoherence of the spin sector [in contrast to the inverse length suppression that was found for adiabatically contacted SCs (Ref. 15)]. We show the exponential decay is due to the contributions of the nonzero modes of the *charge* excitations, which do not enter in the adiabatic case where only the zero modes carry the Josephson current. Finally, we show that the ac Josephson effect also contains important information about spin-incoherent effects in its voltage and temperature dependence.

This paper is organized as follows. In Sec. II we review bosonization for a strongly interacting electron system for energy scales small compared to the charge energy, but arbitrary compare to the spin energy. In Sec. III we study spin pumping and magnetization dynamics in the FM-SILL system. In Sec. IV we study ac and dc Josephson effects in the SC-SILL-SC system. Finally in Sec. V we discuss prospects for experimental realizations of the physics we discussed and directions for future work. Some technical details and results appear in Appendixes A and B.

II. REVIEW OF BOSONIZATION FOR STRONGLY INTERACTING ELECTRONS

In this paper we will be studying a strongly interacting one-dimensional electron system coupled to either a ferromagnet or a superconductor. Strong interactions imply[9] that $E_\sigma \ll E_\rho$. Therefore it is possible to be in a regime of temperatures that may not be small compared to the spin energy E_σ, but are still small compared to the charge energy E_ρ. This means that we are no longer free to use standard bosonization procedures[1–3] for the electron operator because these formulas rely on the assumption that the relevant energy scales on which the system is probed are small compared to both spin and charge energies.

For a strongly interacting system we require a formalism that applies to all energy scales relative to the spin energy (and in particular to $k_B T \gtrsim E_\sigma$), but only low-energy scales relative to the charge energy. Such a formalism was recently developed by Matveev *et al.*[16] and applied to the evaluation

of the spectral function and tunneling into a strongly interacting electron system at arbitrary temperatures relative to the spin energy.[17] In this section we review the essential elements of their work and show how to represent the electron operator in a strongly interacting one-dimensional electron system. The general expressions for electron correlation functions obtained in this way provide a launching point for numerical studies in the intermediate-temperature (with respect to spin) regime $k_B T \approx E_\sigma$, which is not well handled by existing analytical methods.

A. Basic assumptions and general expressions

As is typical of interacting one-dimensional electron systems, we assume that the Hamiltonian in the strongly interacting case is spin-charge separated,[9,14] $H = H_\rho + H_\sigma$. Here, H_ρ is identical to that of the LL, and is given by Eq. (1). On the other hand, the spin Hamiltonian at arbitrary temperatures is to a very good approximation given by a nearest-neighbor antiferromagnetic Heisenberg spin chain,[14,18,19]

$$H_\sigma = \sum_l J\vec{S}_l \cdot \vec{S}_{l+1}, \tag{2}$$

where evidently the spin energy is set by J: $E_\sigma = J$. The basic idea is to represent the electron operator as a product of operators that describe the holons $\Psi(x)$ (spinless fermions that naturally arise in the context of strongly interacting fermions and the spin-incoherent regime[20,21]) and the spin degrees of freedom $\vec{S}_l$. The holon operators (denoted by $\Psi^\dagger, \Psi$) by construction satisfy the equation

$$\Psi^\dagger(x)\Psi(x) = \psi_\uparrow^\dagger(x)\psi_\uparrow(x) + \psi_\downarrow^\dagger(x)\psi_\downarrow(x), \tag{3}$$

where ψ_s is the electron annihilation operator for electrons of spin projection s, and $\psi_s^\dagger$ is the corresponding electron creation operator.

The issue of how to bosonize the electron operator for a strongly interacting system earlier arose in the context of the large-U limit of the one-dimensional Hubbard model. Penc *et al.*[22] wrote the electron creation operator as

$$\psi_s^\dagger(0) = Z_{0,s}^\dagger \Psi^\dagger(0), \tag{4}$$

where $Z_{0,s}^\dagger$ creates a site on spin chain (2) with spin projection s. Expression (4) can be physically motivated as follows. From Eq. (3) it is clear that the creation of an electron is also accompanied by the creation of a holon. However, electrons also carry spin so there must be a component of the electron operator that also creates spin. This is accomplished by $Z_{0,s}^\dagger$. In general, one has $Z_{l,s}^\dagger$ as the object that adds a new site to the spin chain between $l-1$ and l. While this appears physically intuitive, the expression suffers from the drawback that it does not naturally account for the variation in electron density with position in a real electron gas.[23] Matveev *et al.*[16] showed that a remedy for this issue is to define the position at which the spin site is added to chain (2) in terms of the number of holons to the left of the site,

$$l(x) = \int_{-\infty}^{x} \Psi^{\dagger}(y)\Psi(y)dy. \tag{5}$$

In terms of Eq. (5) the electron creation and annihilation operators are defined as

$$\psi_s^{\dagger}(x) = Z_{l(x),s}^{\dagger}\Psi^{\dagger}(x), \tag{6}$$

$$\psi_s(x) = \Psi(x)Z_{l(x),s}. \tag{7}$$

The operators given above explicitly account for the fact that the spins are attached to electrons, and the formulas are valid at all energy scales. It is perhaps worth noting that even though the Hamiltonian is spin-charge separated, the electron operators are not written as a product of a spin piece and a charge piece because the "spin" pieces $Z_{l(x),s}$ also depend on the electron density via Eq. (5).

B. Bosonizing the holon operators

In writing Eqs. (6) and (7) no assumptions have been made about the energy scale relative to the spin and charge energies. We now restrict our considerations to energies small compared to E_ρ, but arbitrary with respect to E_σ. In this case, we are free to bosonize the holon sector. The Hamiltonian for the holons must then necessarily take the low-energy form[20]

$$H_\rho = v_\rho \int \frac{dx}{2\pi}\left[\frac{1}{K}[\partial_x\theta(x)]^2 + K[\partial_x\phi(x)]^2 \right], \tag{8}$$

where the interaction parameter K of the holon is related to the interaction parameter of the charge sector as[20] $K=2K_\rho$, and the spinless fields θ and ϕ can be related to the holon density as[21,24]

$$\Psi^{\dagger}(x)\Psi(x) = \frac{1}{\pi}[k_F^h + \partial_x\theta(x)], \tag{9}$$

where the holon Fermi wave vector is twice the electron Fermi wave vector,[21] $k_F^h = 2k_F$. The bosonic fields satisfy the commutation relations $[\theta(x), \partial_y\phi(y)] = i\pi\delta(x-y)$.

Since we are interested in low energies with respect to the charge energy, the electron operator may be expanded about the two holon Fermi points at $\pm k_F^h$,

$$\Psi(x) = \Psi_R(x) + \Psi_L(x), \tag{10}$$

where $\Psi_R(x)$ destroys a holon near the right Fermi point and $\Psi_L(x)$ destroys an electron near the left Fermi point. The left and right holon operators are bosonized as

$$\Psi_{R,L}(x) = \frac{1}{\sqrt{2\pi\alpha_c}}e^{-i\phi(x)}e^{\pm i[k_F^h + \theta(x)]}, \tag{11}$$

where α_c is a short-distance cutoff on the order of the interparticle spacing a. Combining the results of Eqs. (5), (7), and (9)–(11), one obtains the bosonized form of the electron annihilation operator for spin s,

$$\psi_s(x) = \frac{e^{-i\phi(x)}}{\sqrt{2\pi\alpha_c}}(e^{i[k_F^h x + \theta(x)]} + e^{-i[k_F^h x + \theta(x)]})Z_{l,s}|_{l=(1/\pi)[k_F^h x + \theta(x)]}, \tag{12}$$

and an analogous expression for the electron creation operator $\psi_s^{\dagger}(x)$. Expression (12), however, it not quite complete as it does not account for the discreteness of the charge of the electron. This can be accomplished by interpreting

$$Z_{l,s}|_{l=(1/\pi)[k_F^h x + \theta(x)]} \rightarrow \sum_l Z_{l,s}\delta\left(\frac{1}{\pi}[k_F^h x + \theta(x)] - l \right), \tag{13}$$

after which the full electron annihilation operator (include both left- and right-moving parts) becomes[16]

$$\psi_s(x) = \frac{e^{-i\phi(x)}}{\sqrt{2\pi\alpha_c}}\int_{-\infty}^{\infty}\frac{dq}{2\pi}z_s(q)e^{i(1+q/\pi)[k_F^h x + \theta(x)]}, \tag{14}$$

where

$$z_s(q) = \sum_{l=-\infty}^{\infty} Z_{l,s}e^{-iql}. \tag{15}$$

The expression for electron annihilation operator (14) is the key result of the this section, obtained earlier by Matveev *et al.*,[16] who also showed that in the limit of small energies compared to E_σ the expression correctly recovers the standard LL formulas for the electron annihilation operator. With Eq. (14) correlation functions involving electron operators can be expressed in terms of the correlation functions of the holon and spin sectors at arbitrary temperatures with respect to E_σ, but small energies compared to E_ρ. See Appendixes A and B for examples of the evaluation of the single-particle Green's function near different types of boundaries using bosonization formula (14) and the boundary conditions to be discussed next.

C. Open and Andreev boundary conditions

In this work we will be primarily interested in two types of boundary conditions on the electron operators: (1) open or "hard wall" boundary conditions appropriate for tunnel junctions and (2) Andreev boundary conditions appropriate for adiabatically contacted (no electron backscattering) superconductors. Other, more general, "intermediate" boundary conditions are possible though less generic.[25] To help orient the reader, in each of the two cases above we will briefly summarize the result appropriate for the LL regime for the purpose of drawing contrast with the strongly interacting, finite-temperature SILL regime. Our discussion of the boundary conditions extends the results of Ref. 16.

1. Open boundary conditions

Open boundary conditions, or hard wall boundary conditions, result in electron waves that are perfectly reflected at the boundary. This implies there is no charge current or spin current through the boundary. For concreteness, let us assume that our boundary is located at $x=0$ with the interacting one-dimensional system living on $x>0$. Then an electron

traveling to the left with spin s will be reflected to a right-moving electron with the same spin s,

$$\psi_{L,s}(0) = e^{-i\eta}\psi_{R,s}(0), \tag{16}$$

with a phase shift η that depends on details of the boundary scattering potential.[25] From the standard LL bozonization formulas (in our convention),

$$\psi_{R/L,s}(x) = \frac{1}{\sqrt{2\pi\alpha_c}}e^{-i\phi_s(x)}e^{\pm i[k_F x + \theta_s(x)]}, \tag{17}$$

we see that Eq. (16) implies that $\theta_s(0) = $const, and therefore also that $\theta_\rho(0) = $const and $\theta_\sigma(0) = $const. In computing correlation functions at the boundary, this means that we must take $\theta_\rho(0)$ and $\theta_\sigma(0)$ to be nonfluctuating quantities, allowing only $\phi_\rho(0)$ and $\phi_\sigma(0)$ to fluctuate.

In the strongly interacting case, reflection condition (16) implies that $\theta(0) = $const, where we have used Eq. (12). Since the spin site l is related to the θ field via Eqs. (9) and (5), we have $l \equiv 0$ at $x = 0$. This implies that any correlation function involving electron operators evaluated at the boundary will depend only on $Z_{l=0,s}$ or its Hermitian conjugate, and the fluctuations of the field ϕ at $x = 0$. We will apply these boundary conditions in a calculation of the single-particle Green's function in Appendix A.

For completeness, below we give the expansions of the bosonic fields for a system of finite length L with open boundary conditions at $x = 0$ and $x = L$ (Refs. 24, 26, and 27):

$$\theta(x) = i\sum_{m=1}^{\infty} \sqrt{\frac{2K_\rho}{m}}\sin\left(\frac{m\pi x}{L}\right)(b_m - b_m^\dagger) + \theta^0(x),$$

$$\phi(x) = \sum_{m=1}^{\infty} \sqrt{\frac{1}{2K_\rho m}}\cos\left(\frac{m\pi x}{L}\right)(b_m + b_m^\dagger) + \Phi, \tag{18}$$

where $\theta^0(x) = \frac{\pi x}{L}N$, $[b_n, b_m^\dagger] = \delta_{nm}$, and $[\Phi, N] = i$. For a semi-infinite system, we take $L \to \infty$ and the discrete sums over m become integrals over momentum $q_m = m\pi/L$.

2. Andreev boundary conditions

In a certain sense the Andreev limit is the opposite limit of hard wall scattering from a SC-M, or SC-LL, or SC-SILL interface. Whereas open boundary conditions imply that an incident electron is perfectly reflected, Andreev boundary conditions imply that an electron is perfectly absorbed by the SC (with a concomitant reflected hole of the opposite spin).[28] However, compared to open boundary conditions (16) the Andreev boundary conditions are more subtle as they involve an energy scale, the superconducting gap Δ, that under many circumstances cannot be taken to be infinitely large relative to the energy scales of interest (such as $k_B T$, or an applied voltage) in the interacting one-dimensional system. One of the most important consequences of a finite Δ is an energy-dependent reflection coefficient, which ultimately leads to the proximity effect in the normal material.[29] In the context of interacting one-dimensional systems, Andreev boundary conditions have been discussed by a number of authors.[25,30–35] The conclusion of these works is that when a

left-moving electron with spin s is reflected as a right-moving hole with the opposite spin, there is an energy-dependent phase shift $e^{iq\xi}$ (proportional to the momentum difference q with respect to the Fermi point)[33] that multiplies a factor[30] $e^{i\chi}$ that encodes the phase χ of the superconducting order parameter (assumed nonzero for $x < 0$),[25]

$$\psi_{L,s}(0) = (-1)^{f(s)}ie^{i\chi}e^{iq\xi}\psi_{R,-s}^\dagger(0), \tag{19}$$

where $\xi \propto 1/\Delta$ is the superconducting coherence length. The function $f(s) = 0$ for $s = \uparrow$ and $f(s) = 1$ for $s = \downarrow$. Boundary conditions (19) are valid only at energy scales much smaller than Δ, which we will assume throughout this work.

Applying boundary conditions (19) to the LL case [with bosonized electron operator below Eq. (16)] gives $\theta_\sigma \propto \theta_s - \theta_{-s} = $const and $\phi_\rho \propto \phi_s + \phi_{-s} = $const. Therefore, we find that much like the situation of perfect reflection there is no spin current through the interface, but there is a net charge current. Moreover, analysis of singlet superconductivity and spin-density wave correlation functions using Eq. (19) in the LL regime shows that there are suppressed spin fluctuations near (distances less than ξ) the interface.[33] It is perhaps worth noting that if the SC-LL interface has a very weak electron backscattering, the interactions in the LL tend to renormalize the interface scattering.[25,32,34]

For strongly interacting electrons boundary condition (19) implies $\phi(0) = $const and $\Sigma_l Z_{l,s}\delta[\theta(0)/\pi - l] = \Sigma_l Z_{l,-s}^\dagger\delta[\theta(0)/\pi - l] \Rightarrow Z_{l,s} = Z_{l,-s}^\dagger$ at the interface, where we have again used Eq. (12) and also Eq. (13). We show in Appendix B that we recover our earlier results[15] for SC-SILL junctions in the Andreev limit using this formalism.

Finally, for completeness and for unification of our notation, below we give the expansions of the bosonic fields[15,30,36] for Andreev boundary conditions on a system of length L (see Fig. 3) with superconducting order-parameter phase difference $\chi = \chi_1 - \chi_2$:

$$\theta(x) = \sum_{m=1}^{\infty} \sqrt{\frac{2K_\rho}{m}}\cos\left[\frac{m\pi}{L}\left(x + \frac{\xi}{2}\right)\right](b_m + b_m^\dagger) + \theta^0,$$

$$\phi(x) = i\sum_{m=1}^{\infty} \sqrt{\frac{1}{2K_\rho m}}\sin\left[\frac{m\pi}{L}\left(x + \frac{\xi}{2}\right)\right](b_m - b_m^\dagger) + \Phi(x), \tag{20}$$

where $\Phi(x) = \pi(J' + \frac{\chi}{\pi})\frac{x}{L}$, $[b_n, b_m^\dagger] = \delta_{nm}$, and $[\theta^0, J'] = i$. The topological number $J' = (N_\uparrow + N_\downarrow)/2 + 1$ and the total spin of the system $M \equiv (N_\uparrow - N_\downarrow)/2$ must satisfy the constraint $J' + M = $even.[30] We have also included the proximity effects[33] via the length ξ in the field expansions that appeared earlier in general expression (19) for the boundary conditions appropriate to Andreev reflection.[37]

Note that compared to the field expansions for open boundary conditions (18), the expansions for Andreev boundary conditions (20) have θ and ϕ "switched." This can be understood in simple physical terms: θ and ϕ are conjugate fields so if one is a constant, the other is strongly fluctuating. Thus, the switching of the fields comes from "opposite" natures of the two boundary conditions in the charge sector. For open boundary conditions there is no charge cur-
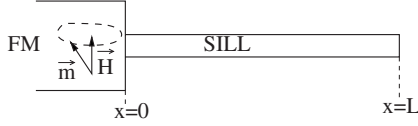
FIG. 1. Schematic of the model we study. A ferromagnet is tunnel coupled at $x=0$ to a spin-incoherent Luttinger liquid of length L. As the magnetization vector $\vec{m}$ precesses about an effective magnetic field $\vec{H}$, spin is pumped into the SILL. Spin accumulation in the SILL will lead to renormalization of the magnetization dynamics.

rent through the boundary, while for Andreev boundary conditions the charge current on either side of the boundary is unchanged by its presence because there is no electron backscattering there. Had we been concerned with the bosonization of the spin sector, we would have found that due to the absence of spin current through the boundary, that sector would have expansions appropriate for open boundary conditions similar to Eq. (18).[15,30,36]

Having spent the time to develop the formalism used in our calculations, we now turn our attention squarely to the physics of hybrid junctions involving spin-incoherent Luttinger liquids. We first discuss a junction consisting of a ferromagnet tunnel coupled to a spin-incoherent Luttinger liquid.

III. FM-SILL TUNNEL JUNCTIONS

A. Spin pumping

We consider the setup shown schematically in Fig. 1, whereby a ferromagnet is coupled via a tunnel junction to a spin-incoherent Luttinger liquid. A similar situation was considered by Bena and Balents[38] in the context of a FM-LL junction. They found that when the magnetization $\vec{m}$ of the FM acquires a time dependence (perhaps by the application of external fields), spin current is pumped into the adjoined LL at the rate

$$\langle \vec{I}(t) \rangle = - A_1 \vec{m} \times \frac{d\vec{m}}{dt} + A_2 \frac{d\vec{m}}{dt}, \quad (21)$$

where A_1 and A_2 are temperature-dependent parameters depending on the interface tunneling and interactions in the LL. Below we show that an identical expression is obtained for a SILL and we derive explicit expressions for A_1 and A_2 in this case. We schematically indicate how the temperature dependence crosses over from the LL to the SILL case. We note that Eq. (21) was earlier derived for a noninteracting metal attached to a FM. In this case, A_1 and A_2 are temperature independent to lowest order.[39–41] Thus, the temperature dependence of A_1 and A_2 encodes information about the interactions and may also indicate whether the attached 1-d system is in the LL or SILL regime. We now turn to a derivation of these results.

We assume that the ferromagnet is itinerant, such as Fe or Co, and can be described by a Stoner-type model in which there is a different density of states for spin-up and spin-down electrons. The different density of states can be ab-

sorbed into distinct effective tunneling matrix elements t_s for spin-up and spin-down electrons.[42,43] This also implies that the local action[44] describing the spin-up and spin-down electrons are identical to that of noninteracting electrons,

$$S_{\text{FM}} = \frac{1}{\beta} \sum_{\omega_n} \sum_{s=\uparrow,\downarrow} \frac{|\omega_n|}{2\pi} |\phi_m^s(\omega_n)|^2, \quad (22)$$

where β is the inverse temperature and $\phi_m^s(\omega_n)$ are the bosonic fields describing local fluctuations at the tunneling point, $x=0$, of spin-up and spin-down electrons in the FM. For the SILL we have the following spin-charge separated form of the local action:

$$S_{\text{SILL}} = \frac{1}{\beta} \sum_{\omega_n} \frac{K_\rho |\omega_n|}{2\pi} |\phi^\rho(\omega_n)|^2 + S_{\text{SILL}}^\sigma, \quad (23)$$

where $0 < K_\rho < 1$ is the Luttinger parameter of the charge sector of the SILL, $\phi^\rho = (\phi^\uparrow + \phi^\downarrow)/\sqrt{2}$ is the bosonic charge field, and S_{SILL}^σ is the action for the spin sector which will actually play no role in the evaluation of correlation functions in the spin-incoherent regime as it effectively drops out, leading to "superuniversal spin physics."[9]

For an arbitrary spin-quantization axis relative to the magnetization $\vec{m}$, it is useful to define

$$\hat{u}_\pm = (1 \pm \hat{m} \cdot \vec{\sigma})/2, \quad (24)$$

which projects the spin-quantization axis onto the magnetization direction $\hat{m}$. Here $u_\pm$ is a 2×2 matrix and $\vec{\sigma}$ are the Pauli spin matrices. The tunneling Hamiltonian then takes the form[43,45]

$$H_{\text{tun}}^{\text{gen}} = F^\dagger W \psi + \psi^\dagger W^\dagger F, \quad (25)$$

where F ($F^\dagger$) annihilates (creates) an electron in the FM and ψ ($\psi^\dagger$) annihilates (creates) an electron in the SILL. The tunneling is assumed to occur at $x=0$ as shown in Fig. 1. The 2×2 tunneling matrix W is then

$$W = \sum_{s=\pm} t_s \hat{u}_s. \quad (26)$$

Tunneling Hamiltonian (25) can be expressed more explicitly as

$$H_{\text{tun}} = \sum_s u_1 F_s^\dagger \psi_s + \vec{m}(t) \cdot \sum_{s,s'} u_2 F_s^\dagger \frac{\vec{\sigma}_{s,s'}}{2} \psi_{s'} + \text{H.c.}, \quad (27)$$

where $\vec{m}(t)$ is the time-dependent magnetization of the ferromagnet, and $u_1 = (t_+ + t_-)/2$ and $u_2 = (t_+ - t_-)/|\vec{m}|$. (From here onward we will assume $|\vec{m}| = 1$.) The spin current operator is obtained from the relation $\vec{I}(t) = \frac{d\vec{M}}{dt} = -\frac{i}{\hbar}[\vec{M}, H_{\text{tun}}]$, where $\vec{M} = \frac{1}{2}\psi_s^\dagger \vec{\sigma}_{ss'} \psi_{s'}$ is the spin density in the SILL at the boundary and summation over repeated spin indices is understood. It follows then that the spin current operator is given by

$$\vec{I}(t) = \frac{iu_1}{2} F_s^\dagger \vec{\sigma}_{ss'} \psi_{s'} + \frac{iu_2}{4} \vec{m} F_s^\dagger \psi_s + \frac{u_2}{4} F_s^\dagger \vec{m} \times \vec{\sigma}_{ss'} \psi_{s'} + \text{H.c.}$$

(28)

The spin current, $\langle \vec{I}(t) \rangle = -\frac{i}{\hbar} \int dt' \Theta(t-t') \langle [I(t), H_{\text{tun}}(t')] \rangle$, is obtained[38] from second-order time-dependent perturbation theory which gives

$$\langle \vec{I}(t) \rangle = \vec{m}(t) \text{Im}(u_2^* u_1) \text{Re}[C(0)] - \int \frac{d\omega}{h} C(\omega) e^{-i\omega t}$$
$$\times \left(\text{Im}(u_2^* u_1) \vec{m}(\omega) + \frac{|u_2|^2}{2} \vec{m}(\omega) \times \vec{m}(t) \right), \quad (29)$$

where $C(\omega)$ is the Fourier transform of the retarded Green's function $C(t-t') = -i\Theta(t-t') \langle [F_s^\dagger(t)\psi_s(t), \psi_s^\dagger(t')F_s(t')] \rangle$. If we assume that the typical frequencies of the magnetization precession are in the gigahertz range[41] (which corresponds to energies of roughly 100 mK) or smaller, then this is a small energy scale in the problem and we may safely expand $C(\omega)$ for small ω. It can be easily checked that the zero-frequency terms cancel exactly, leaving only the contributions linear in ω, provided we drop terms proportional to ω^2 and all higher powers. Upon integration over frequency, the linear frequency terms are converted to time derivatives yielding expression (21) where the coefficients A_1 and A_2 are given by

$$A_1 = -\frac{i}{h} C'(0) \frac{|u_2|^2}{2}, \quad A_2 = -\frac{i}{h} C'(0) \text{Im}(u_2^* u_1). \quad (30)$$

Since only the imaginary part of $C(\omega)$ is odd with respect to frequency, only this piece will contribute to A_1, A_2 and those parameters will therefore be real quantities. The final step is to compute the temperature dependence of $C'(\omega=0)$ in the spin-incoherent regime.

Since we are working within linear response, the commutator in $C(t) = -i\Theta(t) \langle [F_s^\dagger(t)\psi_s(t), \psi_s^\dagger(0)F_s(0)] \rangle$ is evaluated in the state where there is no tunneling between the ferromagnet and the SILL, and a hard wall at $x=0$. By writing $C(t) = -i\Theta(t)[\langle F_s^\dagger(t)F_s(0)\rangle \langle \psi_s(t)\psi_s^\dagger(0)\rangle - \langle F_s(0)F_s^\dagger(t)\rangle \langle \psi_s^\dagger(0)\psi_s(t)\rangle]$, and noting that the long-time behavior of both terms has the same functional dependence, we can easily extract the long-time behavior[9] of $C(t)$ using Eqs. (22) and (23). (A more careful calculation that yields the same result is given in Appendix A.) At finite temperatures (small with respect to the Fermi energy of the ferromagnet and charge energy of the SILL, but large compared to $\hbar v_\rho/L$), we have $\langle F_s^\dagger(t)F_s(0)\rangle \sim \frac{(\pi k_B T/\hbar)}{\sinh(\pi k_B T t/\hbar)}$ and $\langle \psi_s(t)\psi_s^\dagger(0)\rangle \sim \left[\frac{(\pi k_B T/\hbar)}{\sinh(\pi k_B T t/\hbar)}\right]^{1/(2K_\rho)}$. The crucial difference with the Luttinger liquid is that the exponent for the correlations $\langle \psi_s(t)\psi_s^\dagger(0)\rangle$ have changed: in the case of a LL $1/(2K_\rho)$ is replaced by $(1/K_\rho+1)/2$. The remainder of the computations carry through exactly as they would for a LL and we find for $\hbar\omega \ll k_B T$

$$\text{Im}[C(\omega)] \propto \hbar\omega(k_B T)^{\delta^{\text{SILL}}}, \quad (31)$$

where $\delta^{\text{SILL}} = \frac{1}{2K_\rho} - 1$. This implies $C'(0) \propto T^{\delta(T)}$ quite generally so that
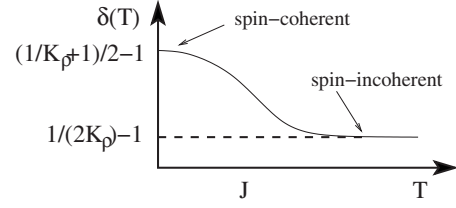

FIG. 2. Temperature dependence of the exponent δ appearing in parameters (32) that describe pumped spin current (21) due to time-dependent motion of a magnetization vector near an interacting acting one-dimensional system at finite temperature. For temperatures T less than spin exchange $J = E_\sigma$, the characteristic spin pumping temperature dependence in the spin-incoherent regime crosses over to the Luttinger liquid form computed in Ref. 38. Note that in the SILL, $\delta(T) < 0$ for $K_\rho > 1/2$ and the qualitative temperature dependence of the pumped spin current is remarkably different from a LL.

$$A_1 \propto T^{\delta(T)} \frac{|u_2|^2}{2}, \quad A_2 \propto T^{\delta(T)} \text{Im}(u_2^* u_1), \quad (32)$$

where $\delta(T)$ interpolates between the LL and SILL regimes as the temperature is swept through the spin energy $E_\sigma \approx J$. The temperature dependence of the exponent $\delta(T)$ is shown schematically in Fig. 2.

It is interesting to note that $\delta > 0$ for $0 < K_\rho < 1$ in the Luttinger liquid regime, implying that less spin current is pumped as the temperature decreases. On the other hand, as long as the system remains in the spin-incoherent regime the opposite behavior may be obtained if $K_\rho > 1/2$: since $\delta^{\text{SILL}} < 0$ *more* spin current is pumped as the temperature is lowered (for $T \gtrsim J$). This is related to the diverging density of states at the boundary[9,24,46] when $K_\rho > 1/2$. In gated cleaved-edge overgrowth quantum wires it appears possible to lower K_ρ down to values on the order of 1/3 and so it may be possible in the spin-incoherent regime to tune between the positive and negative exponent regimes.[6] This qualitative difference should be easily seen in experiment.

B. Renormalization of magnetization dynamics

Having computed the spin current pumped into the SILL by a time-dependent magnetization vector, it is important to ask how the spin accumulation in the SILL in turn affects the magnetization motion. We address this question by computing the renormalization of the Gilbert damping constant due to the spin flow into the SILL. We again closely follow the notation of Bena and Balents[38] to clearly establish a connection to the LL case. The Landau-Lifshitz-Gilbert equation for magnetization $\vec{m}$ precessing around effective magnetic field $\vec{H}$ is

$$\frac{d\vec{m}}{dt} = -\gamma \vec{m} \times \vec{H} + \alpha_0 \vec{m} \times \frac{d\vec{m}}{dt} - \frac{\gamma}{M_s} \vec{I}, \quad (33)$$

where the spin current $\vec{I} = \vec{I}_0 - \vec{I}_b$ flows [$\vec{I}_0$ is given by Eq. (21) for $\vec{I}_b \equiv 0$ and is nonzero only for time-dependent $\vec{m}$] into the SILL and M_s is the saturation magnetization of the ferromagnet. Here γ is the gyromagnetic ratio which is typically equal

to its free-electron value,[41] $\gamma = 2\mu_B/\hbar$, in transition-metal ferromagnets, and α_0 is the dimensionless Gilbert damping parameter in the absence of the spin current. Its value is typically on the order of 10^{-2}. The spin backflow due to spin accumulation in the interacting 1-d system can be described by boundary conditions on left- and right-moving spin currents and is expressed as[43,45]

$$\vec{I}_b = \frac{\vec{\mu}_s}{\mu_s} I_{\delta(T)}(\mu_s, T) - \frac{K_{\text{exch}}}{4\pi} \vec{\mu}_s \times \vec{m}, \qquad (34)$$

where $\vec{\mu}_s$ is the spin chemical potential in the wire related to the magnetization by $\vec{\mu}_s(x) = \vec{M}(x)/\chi(T)$, with $\chi(T)$ as the (generally temperature-dependent) spin susceptibility, and K_{exch} describes the effective exchange coupling between electrons in the SILL and the FM.[47] The current $I_{\delta(T)}(\mu_s, T)$ arises from the electron-tunneling contribution to the spin current.[48] Note that backscattered spin current (34) contains two terms: (i) a term arising from electron transfer from the FM to the interacting 1-d system and (ii) a second term arising from exchange between the local spin density and the precessing magnetization vector $\vec{m}$. Since $\vec{I} = \vec{I}_0 - \vec{I}_b$ this implies that the spin current itself has a contribution due purely to exchange effects, even in the absence of electron transfer.[45]

Our goal in this section is to express Eq. (33) as

$$\frac{d\vec{m}}{dt} = -\gamma' \vec{m} \times \vec{H} + \alpha \vec{m} \times \frac{d\vec{m}}{dt}, \qquad (35)$$

and determine the renormalized parameters γ' and α. From Eq. (33) it is evident that we must find $\vec{I} = \vec{I}_0 - \vec{I}_b$. We have already computed $\vec{I}_0$ [Eq. (21)] in Sec. III A. Now we must determine $\vec{I}_b$ [Eq. (34)], which depends on $\vec{\mu}_s$ and $I_{\delta(T)}(\mu_s, T)$. We start with $\vec{\mu}_s$, which is a function of the spin dynamics in the interacting 1-d system.

It has earlier been shown that SU(2)-invariant electron backscattering leads to diffusive spin behavior in the weakly interacting regime.[43,49] In the strongly interacting regime where the spin sector may effectively be described by a Heisenberg spin chain coupled to phonon distortions[9,14,16,50] the spin dynamics has also been shown to be diffusive.[51] To be on safe ground with our assumption of diffusive spin behavior, we must implicitly keep the spin-phonon coupling discussed in Refs. 9 and 50. Since this coupling is finite when J is finite, we must not take $J \to 0$ (as we safely could for the single-particle Green's function[21] or the superconducting pair correlations[15]). Under the assumption of finite J in the strongly interacting limit, we see that in both cases of weak and strong interactions, the mean free path $l \sim v_\sigma/T$. Following the arguments[20,52] that the spin-incoherent regime can be approached from "below" ($T < J$), we expect the diffusion length l to saturate at the interparticle spacing a for $T \gtrsim J$. Since the diffusion constant $D_s \sim lv_\sigma$, this implies that the diffusion constant in the SILL becomes independent of temperature and takes the value $D_s \approx av_\sigma$. Note that in the limit $v_\sigma \to 0$, there is no spin diffusion, as the spin excitations cannot propagate in the system. Spin will simply pile up at the boundary of the FM without moving further into the

interacting 1-d system. Unlike our earlier results for the temperature dependence of pumped spin into the SILL, Eq. (32) with $\delta^{\text{SILL}} = 1/(2K_\rho) - 1$, which were valid for vanishing spin velocity, here our results are qualitatively dependent on keeping v_σ finite, although still small enough to be in the spin-incoherent regime. Essentially one must keep the "first correction" in J/T to describe spin transport. As we mentioned above, for correlations such as those described by the single-particle Green's function or superconducting pair correlations, we do not need to keep J finite in order to capture the leading effects of spin-incoherent physics. A full self-consistent calculation capturing the detailed temperature dependence of the spin transport in the presence of spin-phonon coupling is highly desirable.

We also assume that the finite-length SILL is characterized by a spin-flip time τ_{sf} (due to impurities, spin-orbit effects, etc). The diffusion equation for spin in the SILL is then

$$i\omega \vec{\mu}_s(x) = D_s \partial_x^2 \vec{\mu}_s(x) - \tau_{\text{sf}}^{-1} \vec{\mu}_s(x), \qquad (36)$$

with the boundary conditions $\partial_x \vec{\mu}_s = -(\frac{1}{D_s \chi(T)}) \vec{I}$ at $x = 0$ and vanishing spin current $\partial_x \vec{\mu}_s(x) = 0$ at $x = L$. See Fig. 1 for the setup. The solution to this equation is simple to obtain and is given by

$$\vec{\mu}_s(x) = \frac{\cosh[\kappa(L-x)]}{\kappa \sinh[\kappa L]} \left(\frac{1}{D_s \chi(T)} \right) \vec{I}(x=0), \qquad (37)$$

where $\kappa = \sqrt{\frac{i\omega - \tau_{\text{sf}}^{-1}}{D_s}}$. If the precession frequency ω is small compared to the inverse spin-relaxation time, then to a good approximation $\kappa \approx 1/\sqrt{D_s \tau_{\text{sf}}}$. At the boundary $x = 0$ we have $\vec{\mu}_s = \xi \vec{I}$, where $\xi = \coth[\kappa L] \frac{1}{\kappa D_s \chi(T)}$. Note that $\partial_x \vec{\mu}_s(x) \sim e^{-\kappa x} \vec{I}(x=0) \approx e^{-x/\sqrt{D_s \tau_{\text{sf}}}} \vec{I}(x=0)$ so the spin current decays exponentially with distance into the SILL on a length scale set by the product of the diffusion constant and the spin-relaxation time. For a long spin-relaxation time, this length scale can be large compared to the interparticle spacing.

We have now determined all parts of spin backflow (34), except the tunneling current $I_{\delta(T)}(\mu_s, T)$, which we now do. The tunneling current is proportional to the imaginary part of the Fourier-transformed correlation function[43,45] $C(\omega)$ defined below Eq. (29),

$$I_{\delta(T)}(\mu_s, T) \propto |u_1|^2 \text{Im}[C(\mu_s/2)] \propto |u_1|^2 (k_B T)^{\delta(T)+1} \sinh\left(\frac{\mu_s}{4k_B T} \right)$$

$$\times \left| \Gamma\left(1 + \frac{\delta(T)}{2} + i\frac{\mu_s}{4\pi k_B T} \right) \right|^2$$

$$\propto \mu_s |u_1|^2 (k_B T)^{\delta(T)} \left| \Gamma\left(1 + \frac{\delta(T)}{2} \right) \right|^2, \qquad (38)$$

where we have used the results of Appendix A and taken the limit $\mu_s \ll k_B T$. Evidently, the main effect of the spin-incoherent physics is to change the exponent $\delta(T)$ to the spin-incoherent value, $\delta^{\text{SILL}} = 1/(2K_\rho) - 1$, so that we have

$$I_{\delta(T)}(\mu_s, T) \propto \mu_s |u_1|^2 (k_B T)^{1/(2K_\rho)-1} \qquad (39)$$

in the spin-incoherent regime. Let us define $\mathcal{T} \equiv I_{\delta(T)}(\mu_s, T)/\mu_s \propto (k_B T)^{\delta^{\text{SILL}}}$, which is temperature depen-

TABLE I. Temperature dependence of key quantities in LL-FM and SILL-FM hybrid systems. Here c' is a constant that depends on L, $\tau_{\rm sf}$, and D_s.

	δ	D_s	χ	ξ	$\mathcal{T}, A_1, A_2$
SILL	$\frac{1}{2K\rho}-1$	const	$\frac{1}{T}$	T	$T^{\delta^{\rm SILL}}$
LL	$\frac{1}{2}(\frac{1}{K\rho}+1)-1$	$\frac{1}{T}$	const	$\coth(c'\sqrt{T})\sqrt{T}$	$T^{\delta^{\rm LL}}$

dent and μ_s independent. At this point the determination of γ' and α is identical to the LL case.[38] We find

$$\gamma' = \frac{\gamma}{1 + \gamma(B_1 A_2 - B_2 A_1)/M_s}, \quad (40)$$

and

$$\alpha = \frac{\alpha_0 + \gamma(B_1 A_1 + B_2 A_2)/M_s}{1 + \gamma(B_1 A_2 - B_2 A_1)/M_s}, \quad (41)$$

where $B_1 = (1+\xi\mathcal{T})/[(1+\xi\mathcal{T})^2 + (\xi K_{\rm exch})^2/(16\pi^2)]$, $B_2 = (\xi K_{\rm exch})/[(1+\xi\mathcal{T})^2 + (\xi K_{\rm exch})^2/(16\pi^2)]$, and A_1, A_2 are given in Eqs. (30) and (32). The temperature dependence in B_1, B_2 is entirely contained in $\mathcal{T}$ and ξ. See Table I below for a comparison of the LL and SILL regimes.

As with the case of a LL, for a SILL we expect there to be little renormalization of γ' relative to γ. But the smallness of the Gilbert damping α_0 means this may obtain a significant temperature-dependent correction depending on $\delta^{\rm SILL}$.

C. Other FM hybrid structures involving a SILL

One could easily imagine other scenarios such as a FM-SILL-FM junction, a FM-SILL-M junction, or even a FM-SILL-SC junction. However, because the spin transport is generally diffusive, for junctions whose length L is long compared to $\sqrt{D_s \tau_{\rm sf}}$, the two contacts to the SILL will more or less behave independently from the point of view of spin transport. To the extent that the two leads are coupled, it is evident from the discussion of Sec. III B that most of the physics (and corresponding general formulas) present for a LL system also applies to the SILL system only with some modifications in the temperature dependence of parameters appearing in the theory. While this may seem like a somewhat trivial result, it is not. The temperature dependence serves as a means to determine whether spin-incoherent physics is present in the system. In particular, we note quite generally that if $1/2 < K_\rho < 1$ the temperature dependence of many quantities (e.g., $\mathcal{T}$, A_1, A_2, and those derived from them) change *qualitatively* relative to the expectations for a LL. Such qualitative differences should be observable in experiment.

It is worthwhile to step back and emphasize some of the essential differences between the SILL and LL cases. First, we note that the different temperature dependences can be traced to three quantities: (1) the correlation function $C(t)$ defined below Eq. (29), (2) the temperature dependence of the spin susceptibility χ, and (3) the temperature dependence of the spin-diffusion constant D_s. So long as the SILL is tunnel contacted to the FM, the correlation function $C(t)$ will

always appear at lowest order in perturbation theory and carry along with it the characteristic temperature dependence of the SILL. This quantity appears in both the description of the pumped spin current and the magnetization dynamics. On the other hand, when we are specifically interested in how the spin propagates in the SILL [as we saw for backscattered spin current (34)] the diffusion constant D_s enters, and also the spin susceptibility via the Einstein relation for the spin conductivity, $\sigma_s = \chi D_s$. We remark that the spin transport is generically diffusive in the spin-incoherent regime, while it may be either ballistic or diffusive in the LL regime.[43,49] Having now flushed out what we feel are the central considerations and results for FM-SILL hybrid structures, we now turn our attention to SC-SILL hybrid structures.

IV. SC-SILL HYBRID STRUCTURES

Throughout this work, we assume that all energy scales are small compared to the superconducting gap Δ unless explicitly stated otherwise, such as in the limit of a short junction (defined below). We are primarily interested in the case where the superconductor is contacted via a tunnel junction with a spin-incoherent Luttinger liquid.

A. SC-SILL junctions

Let us begin our discussion of SC-SILL hybrid systems by considering the simplest case: where the FM in Fig. 1 is replaced by a SC. Earlier we studied the properties of such a junction in the Andreev limit.[15] We found a number of remarkable properties, including a completely universal (independent of the charge and spin Hamiltonians) tunneling density of states in the spin-incoherent regime.[15] In the opposite limit of a tunnel contact between the SC and SILL, no proximity effects of the SC are felt other than those perturbative in the tunneling[53] (in contrast to the Andreev limit, where the behavior of the pair correlations and tunneling density of states follows directly from the form of the field expansions imposed by the boundary conditions[15]). As the proximity effects are already weak in the Andreev limit [they decay exponentially fast with distance, unlike the power-law decay in the LL (Ref. 33)], we do not pursue the very weak proximity effects in the spin-incoherent regime in the limit of a weak tunneling SC-SILL junction.

On the other hand, it is worthwhile to briefly discuss the behavior expected in opposite limit of nearly perfect Andreev reflection with weak (perturbative) electron backscattering at the SC-SILL interface. In the LL context, this was addressed by Vishveshwara *et al.*[31] in a SC-LL-M system who found dips in the conductance of the junction as a function of gate voltage. These dips correspond to multiple electron reflections off the SC interface and occur at voltages corresponding to integer values, $eV = n\hbar\pi/\tau$, of the inverse electron traversal time, $\tau = 2L/v$, where n is an integer counting the number of traversals in the LL.[31] For a SC-SILL-M system we would expect similar qualitative effects, although with an overall suppression in the conductance.[13–15] Finally, we briefly remark that the conductance per channel of an adiabatic SC-SILL junction should drop to 1/2 the value ex-
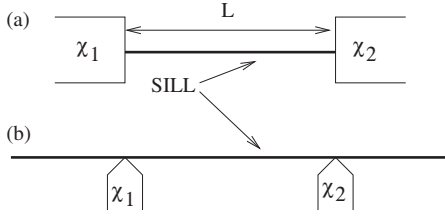
FIG. 3. Schematic of two ways a superconductor can contact a spin-incoherent Luttinger liquid of length L. The phase of the superconducting order parameter at the left contact is χ_1 and at the right contact is χ_2. We assume there is a tunnel barrier between the SC and SILL. In (a) the SILL is "end contacted" and the zero modes (finite length) of the SILL generally come into play, while in (b) the SILL is "bulk contacted" and the zero modes play no role since the system length is effectively infinite.

pected for a SC-LL junction:[33,35] $G_{\text{SC-SILL}} = \frac{1}{2}(2K_\rho)\frac{2e^2}{h}$. Recall that the conductance of the adiabatic SC-LL interface, $G_{\text{SC-LL}} = (2K_\rho)\frac{2e^2}{h}$, generalizes the result for the adiabatic SC-M interface,[54] $G_{\text{SC-M}} = 2\frac{2e^2}{h}$, realized for $K_\rho = 1$.

B. SC-SILL-SC tunnel junctions

We now turn our attention to the final topic of this work, SC-SILL-SC junctions of the types shown in Fig. 3. Our primary focus will be on the behavior of the ac and dc Josephson effects in such a hybrid structure. In the context of SC-LL-SC structures there have already been a number of theoretical works[25,30,36,55–61] and some experimental results.[62–66]

Earlier we studied the dc Josephson effect in a junction such as that shown in Fig. 3(a) only with Andreev boundary conditions at the interfaces of the SILL with the SCs.[15] We found that in spite of the fact that the pair correlations decay exponentially with distance from the boundary of the SC, the Josephson critical current scaled inversely with the length of the SILL and was reduced by a factor of 2 relative to either the SC-LL-SC or SC-M-SC case (which have identical critical currents that also scale inversely with the length).[25,30] We remarked that this "robustness" is essentially related to the fact that the superconductor phase difference only couples to the charge degrees of freedom in the SILL, which remain completely coherent. This effect can also be taken to support the notion[67] that in some sense superconductors are spin-charge separated.[68] Yet another perspective on this result can be obtained by noting that the Andreev boundary conditions on a system of finite length can be mapped onto a system with *periodic* boundary conditions of twice the length.[30] In this way, the Josephson response becomes equivalent to the problem of persistent currents for *spinless* electrons in a ring threaded by magnetic flux, with the superconductor phase difference being simply related to the flux. Therefore, the dc Josephson current for Andreev boundary conditions is essentially given by the physics of *spinless* electrons, which is why spin-incoherent physics does not manifest in a dramatic way. The factor-of-2 reduction in the critical current and Josephson period are a direct consequence of the mapping of the SILL onto spinless electrons.

We now turn our attention to the case of open boundary conditions at the interfaces between the SCs and SILL. In this case phase information must be carried between the two superconductors by Cooper pair tunneling between them. Because the *pairs* must hop from one SC onto the SILL, then onto the other SC, the Josephson current will occur at fourth order in this tunneling process. Moreover, because the Cooper pair tunneling process creates a pair of electrons locally in the SILL near the barrier, many Fourier modes of the electron operator will come into play. This is in contrast to the case of Andreev boundary conditions where only the zero mode enters in the evaluation of Josephson current.[15,30] This crucial difference will turn out to dramatically suppress the Josephson response relative to what one would have for a LL when the SILL is tunnel coupled to SCs. In short, we that find the critical current decays as an exponential of the junction length (compared to a power of the inverse junction length for a LL at zero temperature) with the characteristic decay length as the interparticle spacing.

1. dc Josephson effect

The SC-SILL-SC tunnel junctions shown in Fig. 3 are modeled by the following Hamiltonian: $H = H_{S1} + H_{S2} + H_{\text{SILL}} + H_T$, where $H_{S1/2}$ are s-wave BCS Hamiltonians for the superconductors and H_{SILL} is the sum of Eqs. (8) and (2). The tunneling Hamiltonian H_T is given by

$$H_T = \sum_s T_1 \psi_s^{S1\dagger}(x=0)\psi_s(x=0) + T_2 \psi_s^{S2\dagger}(x=L)\psi_s(x=L) + \text{H.c.}$$

$$(42)$$

We assume that the SCs support a phase difference of $\chi = \chi_2 - \chi_1$ and have the same superconducting gap Δ.

The Josephson current is obtained from the relation

$$\mathcal{J} = -2ek_BT\frac{\partial \ln Z}{\partial \chi},\qquad(43)$$

where $-e$ is the electronic charge and Z is the partition function. We employ imaginary time perturbation theory, where terms dependent on χ appear in fourth order,[36]

$$\ln Z = \int d\tau_1 d\tau_2 d\tau_3 d\tau_4 T_1^2(T_2^*)^2 F_{S1}^\dagger(\tau_1 - \tau_2)$$
$$\times \Pi(0,L;\tau_1,\tau_2,\tau_3,\tau_4)F_{S2}(\tau_3 - \tau_4) + \text{H.c.}$$
$$+ \text{similar terms},\qquad(44)$$

where $\Pi(0,L;\tau_1,\tau_2,\tau_3,\tau_4)$ is a two-particle (cooperon) propagator in the one-dimensional interacting system. The "similar terms" account for time-ordered permutations and spin projections (there are 23 of them, 4! all together). The propagation of Cooper pairs in the superconductors is described by the time-ordered anomalous Green's function F_S,

$$F_{S_i}(\tau - \tau') \equiv \langle T_\tau \psi_\downarrow^{S_i}(\tau)\psi_\uparrow^{S_i}(\tau')\rangle$$
$$= \frac{\pi N_i(0)}{\beta}\sum_n e^{-i\omega_n(\tau - \tau')}\frac{\Delta e^{i\chi_{S_i}}}{\sqrt{\omega_n^2 + \Delta^2}},\qquad(45)$$

where the expectation value is taken with respect to H_{S_i} and

the electron operators in Eq. (45) are evaluated at $x=0$ for S_1 and $x=L$ for S_2.

Of the 24 terms (plus their Hermitian conjugates) that appear in the partition function [Eq. (44)], physical considerations aid us in choosing the most relevant ones depending on the length of the SC separation distance L. There are two important length scales in the junctions we study that give rise to corresponding time scales: the superconducting coherence length ξ and the junction length L.[69] We define "long" to mean $L \gg \xi$ and "short" to mean $L \ll \xi$. We now turn to a discussion of the general expression for $\Pi(0,L;\tau_1,\tau_2,\tau_3,\tau_4)$ in the limit of long and short junctions.

For long junctions the tunneling into the SILL is "fast" and the propagation of the Cooper pairs is "slow." The relevant two-body propagator is

$$\Pi^{\text{long}}(0,L;\tau_1,\tau_2,\tau_3,\tau_4)$$
$$= \langle \psi_s(0,\tau_1)\psi_{-s}(0,\tau_2)\psi_{-s}^{\dagger}(L,\tau_3)\psi_s^{\dagger}(L,\tau_4) \rangle, \quad (46)$$

where ψ_s is the bosonized electron annihilation operator given by Eq. (14) and the averaging is taken with respect to H_{SILL}. We have $|\tau_1-\tau_2| \propto \xi$, $|\tau_3-\tau_4| \propto \xi$, and $|\tau_1-\tau_4| \approx |\tau_2-\tau_3| \propto L$.

In the opposite limit of a short junction $L \ll \xi$, the propagation through the SILL is fast compared to the slow tunneling of the Cooper pairs, so the two-particle propagator approximately separates into a product of two single-particle Green's functions[36]

$$\Pi^{\text{short}}(0,L;\tau_1,\tau_2,\tau_3,\tau_4) \approx \langle \psi_s(0,\tau_1)\psi_s^{\dagger}(L,\tau_2) \rangle$$
$$\times \langle \psi_{-s}(0,\tau_3)\psi_{-s}^{\dagger}(L,\tau_4) \rangle$$
$$\approx G_s(L,\tau_1-\tau_2)G_{-s}(L,\tau_3-\tau_4), \quad (47)$$

where we have $|\tau_1-\tau_2| \propto \xi$, $|\tau_3-\tau_4| \propto \xi$, and $|\tau_1-\tau_3| \approx |\tau_2-\tau_4| \propto L$ [note the reordering of electron operators relative to Eq. (46)].

a. End-contacted SILL when $L \gg \xi$. We first consider the case of an end-contacted SILL, shown in Fig. 3(a). For this geometry expansions (18) of the holon field operators are the appropriate ones. Our assumption that all energy scales are small compared to Δ (specifically $k_B T \ll \Delta$ and $\hbar v_\rho / L \ll \Delta$) enables us to approximate the anomalous Green's function (45) with a delta function in time, $F_{S_i}(\tau-\tau') \approx \pi N_i(0)e^{i\chi_i}\delta(\tau-\tau')$, where $N_i(0)$ is the density of states of the SC S_i at the Fermi energy in its normal state. This approximation forces the times the electrons tunnel into or out of the SILL to coincide, $\tau_1=\tau_2$ and $\tau_3=\tau_4$. Without loss of generality, we let $\tau_3=0$ and define $\tau \equiv \tau_1$. Using the expression for the electron operator [Eq. (14)], propagator (46) reads

$$\Pi^{\text{long}}(0,L;\tau,0) = \left(\frac{1}{2\pi\alpha_c}\right)^2 \sum_{l_1,l_2,l_3,l_4} \int_{-\infty}^{\infty}\frac{dq_1}{2\pi}\int_{-\infty}^{\infty}\frac{dq_2}{2\pi}$$
$$\times \int_{-\infty}^{\infty}\frac{dq_3}{2\pi}\int_{-\infty}^{\infty}\frac{dq_4}{2\pi}e^{-i(q_1l_1+q_2l_2-q_3l_3-q_4l_4)}$$
$$\times \langle e^{-i\phi(0,\tau)}Z_{l_1,s}e^{i(1+q_1/\pi)[k_F^h\cdot 0+\theta(0,\tau)]}$$

$$\times e^{-i\phi(0,\tau)}Z_{l_2,-s}e^{i(1+q_2/\pi)[k_F^h\cdot 0+\theta(0,\tau)]}$$
$$\times e^{-i(1+q_3/\pi)[k_F^h L+\theta(L,0)]}Z_{l_3,-s}^{\dagger}e^{i\phi(L,0)}$$
$$\times e^{-i(1+q_4/\pi)[k_F^h L+\theta(L,0)]}Z_{l_4,s}^{\dagger}e^{i\phi(L,0)} \rangle. \quad (48)$$

The calculation of the Cooper pair propagator is carried out in the same manner as that of the Green's function presented in Appendix A. We first evaluate the expectation value of the spin chain site creation/annihilation operators,

$$\Xi(0,L) \equiv \langle Z_{l_1,s}Z_{l_2,-s}Z_{l_3,-s}^{\dagger}Z_{l_4,s}^{\dagger} \rangle, \quad (49)$$

in the spin-incoherent regime. To this end it is convenient to fully exploit the symmetries and boundary conditions of the problem. We first note that the open boundary conditions at $x=0,L$ force $\theta=$ const at $x=0,L$. As before, we take this constant to be zero so that all θ fields effectively drop out of Eq. (48). The momentum integrals in Eq. (48) can then be done trivially to give delta functions on the sites l_i, which sets $l_1 = l_2 = 0$ and $l_3 = l_4 = k_F^h L / \pi$. Therefore, we have

$$\Xi(0,L) = \langle Z_{0,s}Z_{0,-s}Z_{k_F^h L/\pi,-s}^{\dagger}Z_{k_F^h L/\pi,s}^{\dagger} \rangle, \quad (50)$$

which in the spin-incoherent regime can be evaluated quite simply by appealing to the physics of the spins. Here $\Xi(0,L)$ measures the amplitude for two spins of opposite orientations introduced at $x=L$ to arrive at $x=0$. In general, Eq. (50) can only be evaluated analytically in two limits: (i) the Luttinger liquid limit in which the correlations reduce to the familiar form obtained via standard[1] bosonization techniques (see Ref. 16 for details) and (ii) the spin-incoherent limit. For intermediate temperatures, one must resort to numerical methods. It is not clear to the authors which methods are best suited to their evaluation, but density matrix renormalization group (DMRG) is an obvious candidate. We hope those with expertise will take up the evaluation of such correlation functions which involve removing and adding sites to a spin chain. Numerical evaluation of correlations such as Eq. (50) and simpler versions involving only the addition of one site and the removal of one site (see Appendixes A and B, and Refs. 16 and 17) will take us a long way toward understanding the detailed crossover behaviors with temperature in strongly correlated one-dimensional systems.

Deep in the SILL regime where all the spins are randomized and spin exchange is highly suppressed, one can evaluate correlation function (50) following the physical arguments outlined in Refs. 9 and 21. The basic picture is that when a new site is added to the chain, the other sites shift down one register to accommodate it. When a site is then removed further down the chain. it leaves the "final" chain shifted by one site (for spins in between position of addition and removal) relative to the "initial" chain. In order for the final and initial states to not be orthogonal, all spins in between the addition site and the removal site must be parallel. (Otherwise the states would be orthogonal—see Ref. 9 for further discussion of this point.) This occurs with probability $\left(\frac{1}{2}\right)^N$, where N is the number of sites. The argument is the

same for adding two sites of opposite spin at one location and them removing them at another as in Eq. (50). Expressed in terms of the holon Fermi wave vector and the length L between the initial and final sites, one then has

$$\Xi(0,L) = \left(\frac{1}{2}\right)^{k_F^h L/\pi} = e^{-(k_F^h L/\pi)\ln 2}, \tag{51}$$

from which it follows that

$$\Pi_{\text{end}}^{\text{long}}(0,L;\tau,0) = e^{-(k_F^h L/\pi)\ln 2}\langle e^{-i2\phi(0,\tau)}e^{i2\phi(L,0)}\rangle. \tag{52}$$

The remaining correlations over the charge degrees of freedom can be computed using expansions (18) and the identity $e^{A+B}=e^A e^B e^{-[A,B]/2}$ valid when $[A,B]$ commutes with both A and B. The final result is

$$\Pi_{\text{end}}^{\text{long}}(0,L;\tau,0) = e^{-(k_F^h L/\pi)\ln 2}\left(\frac{\eta}{1+e^{-\eta}e^{-v_\rho \pi\tau/L}}\right)^{1/K_\rho}, \tag{53}$$

where $\eta \ll 1$ is a short-distance cutoff that ensures ultraviolet convergence of the integrals over the charge fluctuations in the wire. Up to unimportant phase factors and other overall multiplicative constants, the Josephson current in the wire is then

$$\mathcal{J}_{\text{end}}^{\text{long}}(\chi) \propto G_1 G_2 e^{-(k_F^h L/\pi)\ln 2}\left(\frac{L}{\alpha_c}\right)\sin(\chi)$$
$$\times \int_0^\pi dx\left(\frac{\eta}{1+e^{-\eta}e^{-x}}\right)^{1/K_\rho}, \tag{54}$$

where the integral has been cut off at times $\tau = L/v_\rho$, the time it takes for charge to propagate between the two ends of the SILL. Compared to the LL case, the most important difference is that the critical current scales as an exponential of the length, $\mathcal{J}_{\text{end}}^{\text{long}}|_{\text{critical}} \propto e^{-(k_F^h L/\pi)\ln 2}\left(\frac{L}{\alpha_c}\right)$, rather than as a power law $\left(\frac{\alpha_c}{L}\right)^{2/K_\rho-1}$ in the LL regime.[55] A measurement of the length dependence of the critical current then serves as clear signature of spin-incoherent physics (or lack thereof), provided one has some knowledge of the density to infer $k_F^h = 2k_F = \pi/a$. We also note that the superconducting phase difference χ appears as an argument to the sine function, rather than the sawtooth form we found for Andreev contacts.[15] The sine form follows from the assumption of tunneling contacts.[25,57,58]

b. Bulk-contacted SILL when $L \gg \xi$. We now consider the case of a bulk-contacted SILL, shown in Fig. 3(b). For this case, we assume that the perturbations induced by the contacts are irrelevant, which is so for $K_\rho > 1/2$.[20] Otherwise, if the perturbations induced by the contacts are relevant, this case reduces to an end-contacted SILL discussed above.

Starting again with Eq. (48) we note that translational symmetry implies that the spin correlations satisfy

$$\Xi(0,L) \equiv \langle Z_{l_1,s}Z_{l_2,-s}Z_{l_3,-s}^\dagger Z_{l_4,s}^\dagger\rangle = \langle Z_{l_1-l_4,s}Z_{l_2-l_3,-s}Z_{0,-s}^\dagger Z_{0,s}^\dagger\rangle, \tag{55}$$

which allows us to shift the sums $\tilde{l}_1 = l_1 - l_4$ and $\tilde{l}_2 = l_2 - l_3$ in Eq. (48). The summation over l_3 leads to a delta function setting $q_2 = q_3$ and the summation over l_4 leads to a delta function setting $q_1 = q_4$. Making the change of variables $q = (q_1+q_2)/2$ and $\tilde{q} = q_1 - q_2$, one can do the integration over $\tilde{q}$ which sets $\tilde{l}_1 = \tilde{l}_2$. The result is

$$\Pi_{\text{bulk}}^{\text{long}}(0,L;\tau,0) = \left(\frac{1}{2\pi\alpha_c}\right)^2 \sum_l \int_{-\infty}^\infty \frac{dq}{2\pi}e^{-i2ql}\Xi(0,L)$$
$$\times e^{-i2(1+q/\pi)k_F^h L}$$
$$\times \langle e^{i2\{(1+q/\pi)[\theta(0,\tau)-\theta(L,0)]-[\phi(0,\tau)-\phi(L,0)]\}}\rangle. \tag{56}$$

Expression (56) is valid quite generally under the assumption that the contacts are irrelevant perturbations and do not lead to the end-contacted result described earlier. If we now specialize to the spin-incoherent case, we have

$$\Xi(0,L) = \langle Z_{l,s}Z_{l,-s}Z_{0,-s}^\dagger Z_{0,s}^\dagger\rangle = \left(\frac{1}{2}\right)^{|l|}. \tag{57}$$

Substituting Eq. (57) into Eq. (56) and integrating over momentum,

$$\Pi_{\text{bulk}}^{\text{long}}(0,L;\tau,0) = \left(\frac{1}{2\pi\alpha_c}\right)^2 \sum_l \left\langle \left(\frac{1}{2}\right)^{|l|}\frac{1}{2}\delta\left[l + \frac{k_F^h}{\pi}[L + \theta(L,0)\right.\right.$$
$$\left.\left. - \theta(0,\tau)]\right]e^{i2\{[\theta(0,\tau)-\theta(L,0)]-[\phi(0,\tau)-\phi(L,0)]\}}\right\rangle. \tag{58}$$

Finally recalling that for $L \gg a$ we can replace the sum over l by an integral,[21] we find

$$\Pi_{\text{bulk}}^{\text{long}}(0,L;\tau,0) = \left(\frac{1}{2\pi\alpha_c}\right)^2\frac{1}{2}e^{-(k_F^h L/\pi)\ln 2}\langle e^{-(k_F^h/\pi)[\theta(L,0)-\theta(0,\tau)]\ln 2}$$
$$\times e^{i2\{[\theta(0,\tau)-\theta(L,0)]-[\phi(0,\tau)-\phi(L,0)]\}}\rangle, \tag{59}$$

which can be evaluated (up to phase factors) to give

$$\Pi_{\text{bulk}}^{\text{long}}(0,L;\tau,0) \propto \left(\frac{1}{2\pi\alpha_c}\right)^2 e^{-(k_F^h L/\pi)\ln 2}\left(\frac{\alpha_c^2}{L^2+v_\rho^2\tau^2}\right)^{\gamma_{K_\rho}}, \tag{60}$$

where $\gamma_{K_\rho} = \frac{1}{2K_\rho} - \frac{K_\rho}{2}[(\frac{\ln 2}{\pi})^2 - 4]$. The Josephson current in the wire is then

$$\mathcal{J}_{\text{bulk}}^{\text{long}}(\chi) \propto G_1 G_2 e^{-(k_F^h L/\pi)\ln 2}\left(\frac{\alpha_c}{L}\right)^{2\gamma_{K_\rho}-1}\sin(\chi)$$
$$\times \int_0^\pi dx\left(\frac{1}{1+x^2}\right)^{\gamma_{K_\rho}}, \tag{61}$$

where we have again cut off the integral at times $\tau = L/v_\rho$.

Evidently, the critical current again scales as an exponential of the length, $\mathcal{J}_{\text{bulk}}^{\text{long}}|_{\text{critical}} \propto e^{-(k_F^h L/\pi)\ln 2}(\frac{\alpha_c}{L})^{2\gamma_{K_\rho}-1}$, rather than a power law[55] $(\frac{\alpha_c}{L})^{2/K_\rho-1}$ in the LL regime. This Luttinger liquid result corrects the result originally obtained by Fazio *et al.*[36,56] by taking into account the proximity effect. Comparing the critical currents of the end-contacted case [Eq. (54)] with the bulk-contacted case [Eq. (61)], one sees that the length dependences are not materially different: they only differ in the inconsequential power law that multiplies the exponential.

c. End-contacted SILL when $L \ll \xi$. To compute the Josephson current we must evaluate the single-particle Green's functions that appear in Eq. (47). Since ξ is a property of the SC which we assume is tunnel contacted to the SILL, it is independent of the properties of the SILL. We will further assume $a \ll L \ll \xi$, the opposite limit of the wire length being shorter than the interparticle spacing not well motivated physically. The single-particle Green's functions appearing in Eq. (47) can readily be evaluated following the method of Appendix A. This gives

$$G_s^{\text{end}} = \frac{1}{2\pi\alpha_c}\left(\frac{1}{2}\right)^{k_F^h L}\langle e^{-i[\phi(\tau)-\phi(L)]}\rangle$$

$$= \frac{1}{2\pi\alpha_c}e^{-(k_F^h L/\pi)\ln 2}\left(\frac{\eta}{1+e^{-\eta}e^{-v_\rho\pi\pi/L}}\right)^{1/4K_\rho}, \quad (62)$$

which immediately leads to the Josephson current

$$\mathcal{J}_{\text{end}}^{\text{short}}(\chi) \propto \Delta G_1 G_2 e^{-2(k_F^h L/\pi)\ln 2}\left(\frac{L}{\alpha_c}\right)^2 \sin(\chi)$$

$$\times \left[\int_0^\pi dx\left(\frac{\eta}{1+e^{-\eta}e^{-x}}\right)^{1/4K_\rho}\right]^2, \quad (63)$$

with a critical current $\mathcal{J}_{\text{end}}^{\text{short}}|_{\text{critical}} \propto e^{-2(k_F^h L/\pi)\ln 2}(\frac{L}{\alpha_c})^2$.

d. Bulk-contacted SILL when $L \ll \xi$. For a bulk-contacted SC in the short-wire limit, we again apply formula (47) where the Green's functions are those appropriate for an infinite system, computed earlier in the literature in Refs. 9 and 21. The result is

$$\mathcal{J}_{\text{bulk}}^{\text{short}}(\chi) \propto \Delta G_1 G_2 e^{-2(k_F^h L/\pi)\ln 2}\left(\frac{\alpha_c}{L}\right)^{2\Gamma_{K_\rho}-1} \sin(\chi)$$

$$\times \left[\int_0^\pi dx\left(\frac{1}{1+x^2}\right)^{\Gamma_{K_\rho}}\right]^2, \quad (64)$$

which evidently has the critical current $\mathcal{J}_{\text{bulk}}^{\text{short}}|_{\text{critical}}$ $\propto e^{-2(k_F^h L/\pi)\ln 2}(\frac{\alpha_c}{L})^{2\Gamma_{K_\rho}-1}$.

Comparing the results for bulk- and end-contacted cases, as well as short versus long wires, we find the most important result is how the Josephson critical current scales with the length of the wire. Suppressing the relatively unimportant power-law decay that multiplies the dominant exponential decay, we find

$$\mathcal{J}^{\text{long}}|_{\text{critical}} \propto e^{-(k_F^h L/\pi)\ln 2}, \quad (65)$$

$$\mathcal{J}^{\text{short}}|_{\text{critical}} \propto e^{-2(k_F^h L/\pi)\ln 2}, \quad (66)$$

so that the decay of critical current with the length of the SILL is twice as fast as for a short wire. This originates in the fact that for a short wire electrons propagate independently so the spin incoherence affects each electron independently, rather than a single coherent pair as in the case of the long wire. It is also worth emphasizing that the assumption of tunneling contacts always results in a Josephson current that is proportional to the product of the bare conductances of the two contacts with a sinusoidal dependence on the phase difference of the superconducting order parameter, $\mathcal{J}(\chi) \propto G_1 G_2 \sin(\chi)$. Therefore it is the length dependence of critical current (66) that provides a smoking gun signature of spin incoherence in the dc Josephson effect with tunnel contacts. For Andreev (adiabatic) contacts, the length dependence is identical to those of a LL or noninteracting one-dimensional system.[15] Instead, the SILL physics is manifest clearly in the flux dependence of the Josephson current which takes on a sawtooth form of half the usual period.[15]

2. ac Josephson effect

The ac Josephson effect occurs when there is a finite voltage V across the SC-SILL-SC system. The Josephson phase acquires a time dependence $\dot{\chi} = 2eV$ leading to a Josephson current that oscillates in time. A subgap dissipative current is also induced, but at small voltages this can be estimated to be small.[36] As in the case of the dc Josephson effect, the qualitative features of the ac Josephson effect also depend on whether the wire is short or long, as defined in the dc case.

a. Long-wire case. We have seen that the qualitative features of the dc Josephson current in the spin-incoherent regime for bulk and end contacts are not very different. That is also true of the ac Josephson effect. The ac Josephson current can be computed from[36]

$$\mathcal{J}(t) = 4\pi^2 e v_F^2 \frac{G_1 G_2}{(4e^2)^2}$$

$$\times \text{Re}\left[\sum_\pm \pm e^{\pm i2eVt}\int_0^\infty dt' e^{\mp ieVt'}\Pi(t')\right], \quad (67)$$

where $\Pi(t)$ is the Cooper pair propagator [Eq. (46)] evaluated at real times. The ac Josephson current can be decomposed into sinusoidal and cosinusoidal components,

$$\mathcal{J}(t) = 4\pi^2 e v_F^2 \frac{G_1 G_2}{(4e^2)^2}[J_s \sin 2eVt + J_c \cos 2eVt], \quad (68)$$

where

$$J_s = -\text{Im}\left[\int_0^\infty dt'[e^{-ieVt'}\Pi(t') + e^{+ieVt'}\Pi(t')]\right], \quad (69)$$

$$J_c = \text{Re}\left[\int_0^\infty dt'[e^{-ieVt'}\Pi(t') - e^{+ieVt'}\Pi(t')]\right]. \quad (70)$$

In the Luttinger liquid regime J_s and J_c oscillate with voltage across the junction, as does the amplitude $J_a = \sqrt{J_s^2 + J_c^2}$.[36] The

frequency of the oscillations of J_a with voltage depends on the relative spin and charge velocities. One period occurs when the spin and charge parts differ by 2π, that is, when $eV = 2\pi/(L/v_\sigma - L/v_\rho)$.[36] In the spin-incoherent regime, we have $v_\sigma \to 0$, implying that vanishing voltages will lead to oscillations and they may cease to be observed. The lack of amplitude oscillations will persist into the spin-incoherent regime, which has effectively only one velocity, the charge velocity. On the other hand, if v_σ is not too different from v_ρ (say, $v_\sigma = v_\rho/10$), then one can expect to find a temperature dependence of the voltage oscillations that reveals spin-incoherent physics in a way analogous to Coulomb drag[50] or charge fluctuation noise.[70] At temperatures below the spin energy, there will be amplitude oscillations in the Josephson current as a function of voltage, while for temperatures above the spin energy there will be no such oscillations because the spin mode effectively does not propagate. This "washing out" of the high-frequency (because of the ratio $v_\sigma/v_\rho \ll 1$) oscillations with temperature is the signature of spin-incoherent physics in the ac Josephson effect.

b. Short-wire case. For a short wire, the voltage dependence is independent of the properties of the 1-d system,[36] be it a LL or a SILL. In this case the ac Josephson current is

$$\mathcal{J}(t) = \frac{2}{\pi} K(eV/2\Delta)\mathcal{J}_c(0)\sin(2eVt), \qquad (71)$$

where $\mathcal{J}_c(0)$ is the zero-voltage critical current for the short wire given in Sec. IV B 1, and $K(x)$ is an elliptical integral. Recall that for a short wire, $\mathcal{J}_c(0) \propto e^{-2(k_F^h L/\pi)\ln 2}$. Thus, the ac Josephson effect is only effective at revealing spin-charge separation in the long-wire limit, and there are no new spin-incoherent features that appear relative to the LL aside from the length dependence of the dc critical current that enters in Eq. (71).

V. DISCUSSION

In this paper we have touched on what we believe are some of the most easily observed consequences of spin-incoherent behavior in ferromagnet/spin-incoherent Luttinger liquid and superconductor/spin-incoherent Luttinger liquid junctions. For the case of FM-SILL junctions, we computed the spin current pumped into the SILL as a result of magnetization dynamics and the effect of spin accumulation in the SILL on the parameters governing the magnetization dynamics. We found that for interactions in the SILL with $1/2 < K_\rho < 1$, the temperature dependence of the spin current and magnetization dynamics is *qualitatively different* from the case of a LL and should thus be observable experimentally. Some of the key differences between FM-SILL and FM-LL systems are summarized in Table I. The crossover from FM-LL to FM-SILL in the exponent δ governing the temperature dependence of several key quantities is shown in Fig. 2.

In the case of SC-SILL junctions our results greatly extend those obtained earlier by us.[15] In that earlier work we were concerned only with the case of adiabatic (Andreev) contact of the spin-incoherent Luttinger liquid to the superconductor. Here we have developed those results further and also discussed the opposite limit of tunnel contacts to the SC. In the tunneling limit we have computed the ac and dc Josephson responses in the geometries shown in Fig. 3. We find that in contrast to the case of adiabatic contacts, the tunnel contacts lead to a Josephson critical current that is exponentially suppressed with the length of the SILL region. This difference arises from the fact that in the Andreev limit the dc Josephson effect is determined solely by the zero modes of the Hamiltonian, while for tunnel contacts the nonzero modes generally dominate the response. These nonzero modes enter because in the tunneling process an electron is created locally near the tunnel barrier and thus requires many wave vectors to build its wave packet.

With the aim of providing a general discussion of junctions of ferromagnets or superconductors with a strongly interacting one-dimensional system, we have couched many of our calculations in the recently developed scheme for bosonizing strongly correlated electron systems.[16] This formalism is valid for arbitrary temperatures with respect to the spin energy E_σ, but requires that the temperatures remain small compared to the charge energy E_ρ. In order to adapt that formalism to the systems we discussed here, we extended those results to express open and Andreev boundary conditions in a language valid for arbitrary temperatures with respect to the spin sector. Through several examples, we showed how various correlation functions could be evaluated and verified that in the spin-incoherent regime the results properly reduce to the results obtained using the world-line picture.[21]

One of our primary motivations for using the language of Ref. 16 is to help provide a clear starting point for numerical studies of the spin-incoherent regime and the many interesting (and probably experimentally relevant) crossovers that occur between it and the Luttinger liquid regime. Within this formalism the charge physics can be computed analytically via a standard bosonization scheme, but the spin sector must be addressed numerically for temperatures on the order of the spin energy E_σ. The types of correlation functions that must be computed numerically are those that add and remove a site (or multiple sites) from a spin chain, such as $\langle Z_{l_1,s} Z_{l_2,s}^\dagger \rangle$, that appear in the evaluation of a single-particle (or multiparticle) Green's function. [See Eq. (A4), for example.] We would like to emphasize that numerical studies of strongly interacting one-dimensional electron systems with appreciable temperature compared to the spin energy is an entirely untouched area and is now ripe for investigation.

On the experimental side, the lack of a clear experimental "smoking gun" observation of the SILL remains a key issue to be addressed. However, there are mounting experimental indications we are not far away.[10–12] Our best numerical estimates[9,24] suggest that many low-density quantum wires sit right on the edge of the spin-incoherent regime and perhaps all that is needed is a focused experimental effort to search for its signatures, rather than any key technical breakthrough. One of our aims in this work it to highlight certain classes of hybrid structures where spin-incoherent physics should be observable.

Finally, we would like to close with what we regard as some of the outstanding theoretical issues surrounding the

spin-incoherent Luttinger liquid. Perhaps the main one is the behavior on temperature scales $k_B T \approx E_\sigma$ that we already alluded to above. Related to this is a better understanding of the crossover between the Luttinger liquid and the spin-incoherent Luttinger liquid regimes. Both of these will likely require a numerical attack as there are no obvious analytical methods available to address them. There is also the issue of spin-orbit coupling that has so far received no attention. For very strong spin-orbit coupling is there novel behavior in the regime $E_\sigma \ll k_B T \ll E_{SO}, E_\rho$? The subject of noise in hybrid structures involving a SILL will be discussed in a forthcoming work.[71]

APPENDIX A: EVALUATION OF $C(\omega)$ USING BOSONIZATION FOR STRONGLY INTERACTING ELECTRONS

In this appendix we compute the correlator $C(t)$ and its Fourier transform $C(\omega)$ using the general formalism for bosonization of strongly correlated electrons in one dimension developed by Matveev *et al.*,[16,17] who also applied it to the evaluation of the single-particle Green's function for an infinite system and its Fourier transform. We summarized the main results of the bosonization scheme in Sec. II. The calculation here is for a semi-infinite or finite system with open boundary conditions, and so is different in detail from what has been discussed in Refs. 16 and 17, but the basic elements of the bosonization are the same. This appendix is meant to illustrate clearly in a specific example which parts of correlation functions can be computed analytically at finite temperatures and which pieces in general require methods yet to be developed, or a numerical attack. As was already discussed in Sec. II the chief difficulty lies in computing the correlations at arbitrary temperatures in the spin sector. It is hoped that the details given here will provide a good starting point for those skilled in numerics to enter the study of strongly interacting one-dimensional systems where there is currently *no* quantitative understanding of the regime $k_B T \approx E_\sigma \ll E_\rho$. Given the typical values of T, E_σ, E_ρ present[6,10,11,24,72] in quantum wires, this "intermediate"-temperature regime may turn out to be the most relevant experimentally.

In the evaluation of spin current (21) (pumped from a ferromagnet into an adjoined one-dimensional system coupled via a tunnel junction), the correlation function

$$C(t) = -i\Theta(t) \sum_s \langle [F_s^\dagger(t)\psi_s(t), \psi_s^\dagger(0)F_s(0)] \rangle \quad (A1)$$

arises in lowest (second)–order perturbation theory. It contains the difference in the products of single-particle Green's

functions for both the FM and the interacting one-dimensional system,

$$C(t) = -i\Theta(t) \sum_s [\langle F_s^\dagger(t)F_s(0)\rangle\langle\psi_s(t)\psi_s^\dagger(0)\rangle - \langle F_s(0)F_s^\dagger(t)\rangle$$

$$\times \langle\psi_s^\dagger(0)\psi_s(t)\rangle], \quad (A2)$$

where the brackets $\langle\cdot\rangle$ denote a thermal average computed with the open boundary conditions described in Sec. II C 1. The correlators $\langle F_s^\dagger(t)F_s(0)\rangle$ and $\langle F_s(0)F_s^\dagger(t)\rangle$ can be computed by standard bosonization methods.[1] The result is

$$\langle F_s^\dagger(t)F_s(0)\rangle = i\langle F_s(0)F_s^\dagger(t)\rangle = \frac{i}{2\pi\alpha_c}\frac{(\pi k_B T/\epsilon_c)}{\sinh\left(\dfrac{\pi k_B T t}{\hbar}\right)},$$

$$(A3)$$

where $\epsilon_c = \hbar v_F/\alpha_c$ is a high-energy cutoff.

Our real objects of interest here are the boundary Green's functions $G_s^+(t) \equiv \langle\psi_s(t)\psi_s^\dagger(0)\rangle$ and $G_s^-(t) \equiv \langle\psi_s^\dagger(0)\psi_s(t)\rangle$ evaluated at arbitrary temperature with respect to E_σ, but small compared to E_ρ. To evaluate these we make use of expression (14) for the electron operator. Straightforward substitution gives

$$G_s^+(t) = \frac{1}{2\pi\alpha_c}\int_{-\infty}^\infty \frac{dq_1}{2\pi}\int_{-\infty}^\infty \frac{dq_2}{2\pi}\sum_{l_1,l_2} e^{-i(q_1 l_1 - q_2 l_2)}$$

$$\times \langle e^{i[(1+q_1/\pi)\theta(t)-\phi(t)]}Z_{l_1,s}Z_{l_2,s}^\dagger e^{-i[(1+q_2/\pi)\theta(0)-\phi(0)]}\rangle.$$

$$(A4)$$

Open boundary conditions at $x=0$ prevent fluctuations of the θ field which sets it to a constant, which we take to be 0. With θ set to a constant, the integrals over momentum can be done trivially resulting in δ functions for l_1, l_2 which kill those sums and selects $l_1 = l_2 = 0$. The final result is remarkably simple,

$$G_s^+(t) = \frac{1}{2\pi\alpha_c}\langle Z_{0,s}(t)Z_{0,s}^\dagger(0)\rangle\langle e^{-i[\phi(t)-\phi(0)]}\rangle. \quad (A5)$$

Similar manipulations yield

$$G_s^-(t) = \frac{i}{2\pi\alpha_c}\langle Z_{0,s}^\dagger(0)Z_{0,s}(t)\rangle\langle e^{-i[\phi(t)-\phi(0)]}\rangle, \quad (A6)$$

where we have made explicit the time dependence in the spin correlators involving $Z_{0,s}^\dagger$ and $Z_{0,s}$. The general theoretical challenge is then to compute those correlation functions which involve adding and removing a site from the spin chain, which is an interacting many-body problem analogous to Fermi edge physics[73–75] only it involves knowledge of the Hamiltonian on potentially many energy scales. However, for $E_\sigma \ll E_\rho$ the time evolution of the spin degrees of freedom are very slow compared to the charge degrees of freedom, regardless of temperature, and can be neglected for times $\lesssim \hbar/E_\sigma$. Note that in the limit $E_\sigma \to 0$, the spin dynamics can always be neglected. Furthermore, if $k_B T \gg E_\sigma$, then the spins become essentially noninteracting, and therefore nondynamical. Then the correlation functions simplify considerably. In the spin-incoherent regime one has[17,21,76]

$$\langle Z_{0,s}^\dagger Z_{0,s} \rangle = \frac{1}{2}, \tag{A7}$$

and

$$\langle Z_{0,s} Z_{0,s}^\dagger \rangle = 1, \tag{A8}$$

independent of time which can be straightforwardly generalized to include an applied external magnetic field.[24,46,73,77] The result is

$$\langle Z_{0,s}^\dagger Z_{0,s} \rangle = p_s, \tag{A9}$$

$$\langle Z_{0,s} Z_{0,s}^\dagger \rangle = 1, \tag{A10}$$

where p_s is the probability of having spin projection s. It takes the values $p_\uparrow = 1 - p_\downarrow = \frac{1}{e^{-E_Z/k_B T}+1}$, where E_Z is the Zeeman energy. Note that the correlation functions $\langle Z_{0,s}^\dagger Z_{0,s} \rangle$ and $\langle Z_{0,s} Z_{0,s}^\dagger \rangle$ in the spin-incoherent regime are identical to their values in the infinite system.[14] This is because in the spin-incoherent regime the spins are nondynamical, so the open boundary conditions which tend to suppress fluctuations have essentially no effect on the spins which are rendered nondynamical by $k_B T \gg E_\sigma$.

Finally, we are left to evaluate $\langle e^{-i[\phi(t)-\phi(0)]} \rangle$, which is identical to $\langle F_s^\dagger(t) F_s(0) \rangle$ and $\langle F_s(0) F_s^\dagger(t) \rangle$ only with the exponent changed,

$$\langle e^{-i[\phi(t)-\phi(0)]} \rangle = \frac{(-i)^{1/2K_\rho}}{2\pi\alpha_c} \left[\frac{(\pi k_B T/\epsilon_c')}{\sinh\left(\frac{\pi k_B T t}{\hbar}\right)} \right]^{1/2K_\rho}, \tag{A11}$$

where $\epsilon_c' = \hbar v_\rho / \alpha_c$ is a high-energy cutoff for the charge sector in the SILL.

Combining results (A3), (A7), (A8), and (A11), we find

$$C(t) = -i\Theta(t)(-i)^{1/2K_\rho} \left(\frac{1}{2\pi\alpha_c} \right)^2 \left[\frac{(\pi k_B T/\tilde{\epsilon}_c)}{\sinh\left(\frac{\pi k_B T t}{\hbar}\right)} \right]^{\delta^{\mathrm{SILL}}+2}, \tag{A12}$$

where $\delta^{\mathrm{SILL}} = 1/2K_\rho - 1$ is given in Table I, and $\tilde{\epsilon}_c$ is an effective high-energy cutoff given by $\tilde{\epsilon}_c = [\epsilon_c (\epsilon_c')^{1/2K_\rho}]^{2K_\rho/(2K_\rho+1)}$. For making the following formulas more compact, we define $\tilde{\delta} \equiv \delta^{\mathrm{SILL}} + 2$.

We are now ready to compute the Fourier transform of Eq. (A12), $C(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t} C(t)$. Making the change of variables $X = \frac{\pi k_B T}{\hbar} t$, we have

$$C(\omega) = \frac{(-i)^{1+1/2K_\rho}}{(2\pi\alpha_c)^2} \left(\frac{\hbar}{\pi k_B T} \right) \left(\frac{\pi k_B T}{\tilde{\epsilon}_c} \right)^{\tilde{\delta}} \int_0^\infty dX \frac{e^{i(\hbar\omega/\pi k_B T)X}}{[\sinh(X)]^{\tilde{\delta}}}. \tag{A13}$$

The integral is standard,

$$\int_0^\infty dX \frac{e^{i(\hbar\omega/\pi k_B T)X}}{[\sinh(X)]^{\tilde{\delta}}} = 2^{\tilde{\delta}-1} \Gamma(1-\tilde{\delta}) \frac{\Gamma\left(\frac{\tilde{\delta}}{2} - i\frac{\hbar\omega}{2\pi k_B T} \right)}{\Gamma\left(1 - \frac{\tilde{\delta}}{2} - i\frac{\hbar\omega}{2\pi k_B T} \right)}. \tag{A14}$$

The result of the integral can be transformed to a more convenient form using the following identities for complex number z: $\pi = \sin(\pi z)\Gamma(z)\Gamma(1-z)$ and $\Gamma(z)^* = \Gamma(z^*)$. This gives

$$\frac{\Gamma\left(\frac{\tilde{\delta}}{2} - i\frac{\hbar\omega}{2\pi k_B T} \right)}{\Gamma\left(1 - \frac{\tilde{\delta}}{2} - i\frac{\hbar\omega}{2\pi k_B T} \right)} = \left| \Gamma\left(\frac{\tilde{\delta}}{2} + i\frac{\hbar\omega}{2\pi k_B T} \right) \right|^2$$

$$\times \sin\left[\pi\left(\frac{\tilde{\delta}}{2} + i\frac{\hbar\omega}{2\pi k_B T} \right) \right] \frac{1}{\pi}, \tag{A15}$$

where the sine can be expanded to pick out the real and imaginary parts. Selecting the imaginary part of $C(\omega)$ we find

$$\mathrm{Im}[C(\omega)] \propto \frac{1}{(2\pi\alpha_c)^2} \left(\frac{\hbar}{\pi k_B T} \right) \left(\frac{\pi k_B T}{\tilde{\epsilon}_c} \right)^{\tilde{\delta}} \sinh\left(\frac{\hbar\omega}{2\pi k_B T} \right)$$

$$\times \left| \Gamma\left(\frac{\tilde{\delta}}{2} + i\frac{\hbar\omega}{2\pi k_B T} \right) \right|^2, \tag{A16}$$

which for $\hbar\omega \ll k_B T$ gives

$$\mathrm{Im}[C(\omega)] \propto \hbar\omega(k_B T)^{\delta^{\mathrm{SILL}}}, \tag{A17}$$

in agreement with our Eq. (31), and the general analytical form earlier obtained by Bena and Balents[38] up to the exponent describing the temperature dependence which is changed in the spin-incoherent regime, as illustrated in Fig. 2.

The calculation above is valid provided that the temperature is larger than the level spacing of the finite-length wire: $k_B T \gtrsim \hbar v_\rho / L$. For $k_B T \lesssim \hbar v_\rho / L$ the finite-length results of Ref. 26 can readily be generalized to finite temperatures. No simple power laws emerge, but rather a more complicated dependence involving hyperbolic trigonometric functions.

APPENDIX B: SC-SILL JUNCTIONS IN THE ANDREEV LIMIT

In this appendix we show that the bosonization scheme discussed in Sec. II recovers our earlier results for the single-particle Green's function, the tunneling density of states, and the pair correlations.[15]

1. Single-particle Green's function

We are interested in computing the Fourier transform of the single-particle Green's function $G_s^+(x,x',t) = \langle \psi_s(x,t)\psi_s^\dagger(x',0) \rangle$, where here x and x' measure the dis-

tance from the boundary with the SC. Taking $x=x'$ and using general formula (14) we have

$$G_s^+(x,t) = \frac{1}{2\pi\alpha_c} \int_{-\infty}^{\infty} \frac{dq_1}{2\pi} \int_{-\infty}^{\infty} \frac{dq_2}{2\pi} \sum_{l_1,l_2} e^{-i(q_1 l_1 - q_2 l_2)} \langle Z_{l_1,s} Z_{l_2,s}^\dagger \rangle$$

$$\times \langle e^{i[(1+q_1/\pi)[k_F^h x + \theta(x,t)] - \phi(x,t)]}$$

$$\times e^{-i[(1+q_2/\pi)[k_F^h x + \theta(x,0)] - \phi(x,0)]} \rangle, \tag{B1}$$

where the θ and ϕ fields have the expansion given in Eq. (20). We take the limit $L \to \infty$ as we did in Ref. 15 so that the zero modes play no role. Since the holon sector is described by Gaussian theory (8) we can make use of the identity $\langle e^{iA} \rangle = e^{-\langle A^2 \rangle/2}$ for operator A and evaluate the correlation functions at zero temperature with respect to the charge energy but infinite temperature with respect to the spin energy: $E_\rho \gg k_B T \to 0 \gg E_\sigma \to 0$. That is, we take both the spin energy and the temperature to zero but always maintain $k_B T \gg E_\sigma$.[21] In the spin-incoherent regime, the spin correlations are translationally invariant so we have $\langle Z_{l_1,s} Z_{l_2,s}^\dagger \rangle = \langle Z_{l_1-l_2,s} Z_{0,s}^\dagger \rangle = 1/2^{|l_1-l_2|}$. Making a change of variables $l = l_1 - l_2$, the summation over the sites of the spin chain results in a delta function, $2\pi\delta(q_1 - q_2)$, which immediately kills one of the momentum integrals and sets $q_1 = q_2$. We relabel the remaining momentum variable q,

$$G_s^+(x,t) = \frac{1}{2\pi\alpha_c} \int_{-\infty}^{\infty} \frac{dq}{2\pi} \sum_{l=-\infty}^{\infty} \langle 2^{-|l|} e^{-iql} e^{i[(1+q/\pi)\theta(x,t) - \phi(x,t)]}$$

$$\times e^{-i[(1+q/\pi)\theta(x,0) - \phi(x,0)]} \rangle. \tag{B2}$$

Next, we define a variable $Y = 1 + q/\pi$ and re-express the integration in terms of this variable,

$$G_s^+(x,t) = \frac{1}{2\pi\alpha_c} \int_{-\infty}^{\infty} \frac{dY}{2} \sum_{l=-\infty}^{\infty} \langle 2^{-|l|}(-1)^l e^{-il\pi Y}$$

$$\times e^{i[Y\theta(x,t) - \phi(x,t)]} e^{-i[Y\theta(x,0) - \phi(x,0)]} \rangle. \tag{B3}$$

Finally, noting that $\delta\{\pi l - [\theta(x,t) - \theta(x,0)]\} = \int_{-\infty}^{\infty} \frac{dY}{2\pi} e^{-i\{\pi l - [\theta(x,t) - \theta(x,0)]\}Y}$ we see that Eq. (B3) correctly reproduces Eq. (4) of Ref. 15 after we recall the relations $\phi_\rho/\sqrt{2} = \phi$ and $\sqrt{2}\theta_\rho = \theta$.[9,20] The identical result therefore follows for the Green's function and the tunneling density of states derived from it.

2. Pair correlations

Here we compute the pair correlation, $F(x) = -\langle \psi_\uparrow(x) \psi_\downarrow(x) \rangle$, a distance x from the superconductor boundary. By direct substitution of Eqs. (14) and (15) we have

$$F(x) = -\frac{1}{2\pi\alpha_c} \int_{-\infty}^{\infty} \frac{dq_1}{2\pi} \int_{-\infty}^{\infty} \frac{dq_2}{2\pi} \sum_{l_1,l_2} e^{-i(q_1 l_1 - q_2 l_2)}$$

$$\times \langle e^{-i\phi(x)} Z_{l_1,\uparrow} e^{i[(1+q_1/\pi)[k_F^h x + \theta(x)] - \phi(x)]}$$

$$\times e^{-i\phi(x)} Z_{l_2,\downarrow} e^{i[(1+q_2/\pi)[k_F^h x + \theta(x)] - \phi(x)]} \rangle. \tag{B4}$$

At the boundary, we showed in Sec. II C 2 that Andreev reflection implies that $Z_{l,\uparrow} = Z_{l,\downarrow}^\dagger$. Since $l = \theta(x=0)$ at the boundary may fluctuate, we expect also that the result $Z_{l,\uparrow} = Z_{l,\downarrow}^\dagger$ approximately holds near (within ξ) of the boundary. Hence,

$$F(x) \approx -\frac{1}{2\pi\alpha_c} \int_{-\infty}^{\infty} \frac{dq_1}{2\pi} \int_{-\infty}^{\infty} \frac{dq_2}{2\pi} \sum_{l_1,l_2} e^{-i(q_1 l_1 - q_2 l_2)}$$

$$\times \langle e^{-i\phi(x)} Z_{l_1,\uparrow} e^{i[(1+q_1/\pi)[k_F^h x + \theta(x)]]}$$

$$\times e^{-i\phi(x)} Z_{l_2,\uparrow}^\dagger e^{i[(1+q_2/\pi)[k_F^h x + \theta(x)]]} \rangle. \tag{B5}$$

We now assume that we are in the spin-incoherent regime where the spin correlations are translationally invariant: $\langle Z_{l_1,s} Z_{l_2,s}^\dagger \rangle = \langle Z_{l_1-l_2,s} Z_{0,s}^\dagger \rangle = 1/2^{|l_1-l_2|}$. Then repeating the steps used above to compute the single-particle Green's function (changing the summation variable $l = l_1 - l_2$ and killing the momentum integral with δ function), we find

$$F(x) \approx -\frac{1}{2\pi\alpha_c} \int_{-\infty}^{\infty} \frac{dq}{2\pi} \sum_l \langle e^{-iql} 2^{-|l|} e^{i2(q/\pi)[k_F^h x + \theta(x)]} e^{-i2\phi(x)} \rangle. \tag{B6}$$

We then make use of the integral representation of the δ function, $\int_{-\infty}^{\infty} \frac{dq}{2\pi} \sum_l e^{-iq\{l - (2/\pi)[k_F^h x + \theta(x)]\}} = \delta\{l - \frac{2}{\pi}[k_F^h x + \theta(x)]\}$ to obtain

$$F(x) \approx -\frac{1}{2\pi\alpha_c} \sum_l \left\langle 2^{-|l|} \delta\left(l - \frac{2}{\pi}[k_F^h x + \theta(x)]\right) e^{-i2\phi(x)} \right\rangle, \tag{B7}$$

which for $x \gg a, \alpha_c$ can be evaluated by taking the discrete sum to an integral: $\sum_l \to \int dl$. Doing so we recover result (9) of Ref. 15 after again recalling the relations $\phi_\rho/\sqrt{2} = \phi$ and $\sqrt{2}\theta_\rho = \theta$.[9,20]

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